The failure of a conjectured S_4 symmetry in the three-state checkerboard Potts model

1987 J. Phys. A: Math. Gen. 20 L917

(http://iopscience.iop.org/0305-4470/20/14/004)

J. Phys. A: Math. Gen. **20** (1987) L917–L921. Printed in the UK

LETTER TO THE EDITOR

The failure of a conjectured S_4 symmetry in the three-state checkerboard Potts model

I G Enting

Division of Atmospheric Research, CSIRO, Private Bag No 1, Mordialloc, Victoria 3195, Australia

Received 9 June 1987

Abstract. Low-temperature expansions are obtained for the three-state Potts model on the checkerboard lattice. The four Boltzmann weights were set to various integer multiples of a low-temperature variable x. The series were evaluated to order x^{19} but it was found that the conjectured S_4 symmetry failed at order x^{14} for the zero-field partition function, the spontaneous magnetisation and the zero-field susceptibility.

Several years ago, Jaekel and Maillard (1984) noticed that disorder-point solutions for a general q-state Potts model on a checkerboard lattice showed an unexpected S_4 symmetry. In terms of the checkerboard lattice shown in figure 1, four types of interaction (denoted a, b, c and d) occur. The Jaekel–Maillard conjecture is that the partition function for the q-state model is invariant under all permutations of a, b, c and d. Given the various symmetries (C_{4v}) arising from translational and rotational invariance, the question of testing for the full S_4 symmetry reduces to testing for invariance of the partition function under interchanges of a pair of non-parallel interactions such as c and d.

Additional evidence to support the conjectured S_4 symmetry was presented by Maillard and Rammal (1985) who gave a large-q expansion and also considered various

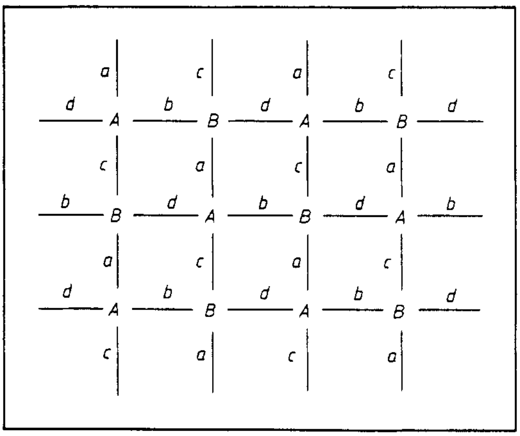

Figure 1. Interactions on the checkerboard lattice showing the two site types, A, B, and the four bond types, a, b, c, d.

known special cases of the model. Additional confirmation is given by the disorder-point solutions for the Ising model susceptibility given by Dhar and Maillard (1985).

The $q=3$ checkerboard Potts model partition function can be written as the limit of a sum over all states of N spin variables τ_i which are located at all sites i of the lattice and which take the values 0, 1 or 2. Thus

$$Z_N = \sum_{\tau_1=0}^{2} \dots \sum_{\tau_N=0}^{2} (ax)^{n_a}(bx)^{n_b}(cx)^{n_c}(dx)^{n_d}\mu^{ns} \tag{1}$$

where n_a, n_b, n_c, n_d are the number of nearest-neighbour bonds (of types a, b, c, d respectively) for which the spin variables at each end of the bond are in different states, and n_s is the number of spin variables not in state 0.

The conjecture of Jaekel and Maillard (1984) is that $Z(a, b, c, d;x, \mu) = \lim_{N\to\infty} Z_N(a, b, c, d;x, \mu)^{1/N}$ is invariant under all permutations of a, b, c and d, not only for the $q=3$ case considered here but for all q. The present letter describes a test that refutes this conjecture. The finite-lattice method of series expansion (see de Neef and Enting (1977) and additional references in § 2 below) is used to construct series expansions for Z in powers of x up to x^{19}, with the μ dependence expanded in powers of $y=1-\mu$ up to y^2. The S_4 symmetry is found to fail at x^{14}.

The finite-lattice method of series expansion is one of a class of techniques which, to paraphrase the words of Wortis (1974), 'substitutes algebraic complexity for combinatorial complexity'. It was first applied to high-temperature expansions for the three-state square-lattice Potts model (de Neef and Enting 1977). Further development of the method has involved the derivation of general closed-form expressions for the powers $\nu(j, k)$ (see (9) below) (Enting 1978a), the generalisation of the method to low-temperature series (Enting 1978b) and the construction of finite-lattice partition functions by adding one site at a time (Enting 1980a). The method has been applied to various Potts model systems (Enting 1980b, c, Enting and Wu 1982, Adler *et al* 1983).

The earlier square-lattice finite-lattice formalism can be readily generalised to rectangular-lattice symmetry. Thus series for checkerboard systems could be obtained by using a two-site unit cell. However, this approach would not be the most efficient in terms of the number of series terms obtained and so a specific generalisation of the finite-lattice method to checkerboard systems is desirable.

The starting point for the finite-lattice method is the existence of a connected-graph expansion for the free energy of any finite graph α

$$f(\alpha) = \sum_{\beta\subseteq\alpha} w(\alpha, \beta)g(\beta) \tag{2}$$

where the irreducible contributions $g(\beta)$ are independent of α and are zero unless β is connected. Each incidence factor $w(\alpha, \beta)$ is the number of ways β can occur as a subgraph of α.

In the limit, as α tends to a large uniform regular lattice of N sites, all the weights tend to N, so long as configurations of sites and bonds that cannot be transformed into one another by translation are regarded as distinct 'graphs'. On non-uniform systems such as the checkerboard lattice, the 'graph-equivalence' criterion must also include the requirement that the translation maps each interaction onto an equal interaction.

The finite-lattice method classifies graphs according to the smallest rectangle within which they can be embedded. On staggered systems with two alternating types of site, A and B, these minimal rectangles give a convenient way of classifying each graph,

α, as type A or B according to the type of site at the top-left corner of the minimal rectangle enclosing α.

Thus for staggered systems, (2) becomes

$$\lim\left(\frac{2}{N}f_N\right)=\sum_{m,n}\left[\left(\sum_{\alpha\in M(m,n;A)}g(\alpha)\right)+\left(\sum_{\alpha\in M(m,n;B)}g(\alpha)\right)\right] \tag{3}$$

where $M(m, n;A)$ is the set of all connected graphs that are enclosed by an $m\times n$ type A rectangle but not by any smaller rectangle. The finite-lattice method truncates (3) as

$$\lim\left(\frac{2}{N}f_N\right)=\sum_{\substack{m,n\\ m+n\leq 2k+1}}\left[\left(\sum_{\alpha\in M(m,n;A)}g(\alpha)\right)+\left(\sum_{\alpha\in M(m,n;B)}g(\alpha)\right)\right]+\delta \tag{4}$$

where, for a suitable expansion variable z, δ is of order z^k or smaller.

Assuming the general applicability of (2) we can write

$$f^A_{mn}+f^B_{mn}=\sum_{m'\leq m}\sum_{n'\leq n}(n-n'+1)(m-m'+1)\left(\sum_{\alpha\in M(m',n';A)}g(\alpha)+\sum_{\alpha\in M(m',n';B)}g(\alpha)\right) \tag{5}$$

where f^A_{mn}, f^B_{mn} are the free energies of $m\times n$ rectangles of types A and B respectively.

While graphs α of the two types A and B contribute unequally to each of f^A_{mn}, f^B_{mn}, only symmetric combinations contribute to the sum $f^A_{mn}+f^B_{mn}$. Therefore the results of Enting (1978a) can be applied to give

$$\sum_{\alpha\in M(m,n;A)}g(\alpha)+\sum_{\alpha\in M(m,n;B)}g(\alpha)$$

$$=\sum_{m'=1}^{m}\sum_{n'=1}^{n}\eta(m',m)\eta(n',n)\left(f^A_{mn}+f^B_{mn}\right) \tag{6}$$

with

$$\eta(i,j)=\begin{cases}1 & i=j\\ -2 & i+1=j\\ 1 & i+2=j\\ 0 & \text{otherwise.}\end{cases} \tag{7}$$

Substituting into (4) gives (following Enting (1978a))

$$\lim\left(\frac{2}{N}f_N\right)=\sum_{\substack{m,n\\ m+n\leq 2k+1}}\nu(m,n)\left(f^A_{mn}+f^B_{mn}\right)+\delta \tag{8}$$

with

$$\nu(m,n)=\begin{cases}1 & m+n=2k+1\\ -3 & m+n=2k\\ 3 & m+n=2k-1\\ -1 & m+n=2k-2.\end{cases} \tag{9}$$

Enting (1978b) showed that the finite-lattice method would generate low-temperature expansions if the f_{mn} were calculated using fixed boundary conditions.

In terms of the definition (1), the error δ in (8) would be of order $x^{4(k+1)}$. For computational purposes it is convenient to take the exponential of (8) to give

$$Z = \lim Z_N^{2/N} \approx \prod_{\substack{m,n \\ m+n\leq 2k+1}} \left(Z_{mn}^A\right)^{\nu(m,n)} \left(Z_{mn}^B\right)^{\nu(m,n)}. \tag{10}$$

The Z_{mn} are calculated by using a transfer-matrix approach that builds up the lattice of width m one site at a time. It is thus desirable to keep m as small as possible.

We use the relations

$$Z_{mn}^A(a, b, c, d) = Z_{mn}^B(c, d, a, b) \tag{11}$$

and

$$Z_{mn}^A(a, b, c, d) = \begin{cases} Z_{nm}^A(b, c, d, a) & m \text{ odd} \\ Z_{mn}^A(d, a, b, c) & m \text{ even.} \end{cases} \tag{12}$$

It is thus possible to write (10) as

$$Z(a, b, c, d) \approx X_1(a, b, c, d) X_2(b, c, d, a) X_1(c, d, a, b) X_2(d, a, b, c) \tag{13}$$

where

$$X_1(a, b, c, d) = \prod_{m=1}^{k} \prod_{n=m}^{2k+1-m} \left(Z_{mn}^A(a, b, c, d)\right)^{\nu(m,n)} \tag{14a}$$

$$X_2(a, b, c, d) = \prod_{m=1}^{k} \prod_{n=m+1}^{2k+1-m} \left(Z_{mn}^A(a, b, c, d)\right)^{\nu(m,n)}. \tag{14b}$$

As noted above, the various finite-lattice partition functions are calculated by building up the rectangle one site at a time. This process is described explicitly by Enting (1980a) for the case of generating closed loops on the square lattice.

In keeping with the product form (13) the test for S_4 symmetry, i.e. comparing $Z(a, b, c, d)$ with $Z(a, b, d, c)$, was performed by constructing the ratio

$$R = \frac{X_1(a, b, c, d) X_2(b, c, d, a) X_1(c, d, a, b) X_2(d, a, b, c)}{X_1(a, b, d, c) X_2(b, d, c, a) X_1(d, c, a, b) X_2(c, a, b, d)} \tag{15a}$$

$$= 1 + \text{terms of order } x^{4(k+1)} \qquad \text{(if } S_4 \text{ symmetry holds).} \tag{15b}$$

The field dependence was expressed in terms of $y = 1 - \mu$ and only the terms involving y^0, y^1 and y^2 were retained. The use of $k = 4$ meant that the series were obtained correct to x^{19}. As in other finite-lattice series calculations, the possible occurrence of large integer coefficients, especially in the intermediate stages, was handled by using residue arithmetic. All arithmetic was handled by taking residues modulo primes p_i with $p_i = 2^{15} - q_i$. Also, as noted previously, the tests were performed for fixed integer values of a, b, c and d. The series were evaluated for $(a, b, c, d) = (1, 2, 3, 5)$ modulo $2^{15} - 19$ and the relation (15b) was found to fail at order x^{14}.

While the single failure is sufficient to refute the conjecture of an S_4 symmetry, a number of further tests were carried out (successfully) in order to verify the correctness of the computer routines.

(i) The tests with weights $(1, 2, 3, 5)$ were carried out with the width k successively set to 1, 2, 3 (at which point the failure was first detected) and 4, in order to check that each time k was increased by 1, series terms calculated using smaller values of k were given correctly. Since the weights $\nu(m, n)$ depend on k (see (9)) this check gives quite a strong test of the requirement that the partition functions Z_{mn} are being evaluated consistently.

(ii) The program was run with weights $(1, 1, 1, 1)$ to ensure that the first two terms in the numerator of $(15a)$ reproduced the known square-lattice Potts model series (Enting 1980b). Various other subproducts in $(15a)$ were checked to ensure that these multiplications were being performed correctly.

(iii) The routines were run with weights $(1, 1, 0, 0)$ for $k=4$. As required, only the trivial contribution of 1 remained when the numerator of $(15a)$ was evaluated.

(iv) The routines were run with weights $(1, 1, 1, 0)$ for $k=4$. As required, the numerator reproduced the low-temperature series for the triangular lattice (with a modified field variable) as given by Enting (1974).

The success of the various tests serves to support the correctness of the general checkerboard-lattice calculations and the conclusion that the conjectured S_4 symmetry fails at order 14 for $q=3$.

After modifying the computer program to perform the same calculations for general q, it was found that for $q=4$ the S_4 symmetry again failed at order 14. For $q=2$ (where only even powers of x occur) the series were evaluated to x^{28} (i.e. $k=6$ was used). The S_4 symmetry for the susceptibility was found to fail at x^{24} while the known S_4 symmetry in the zero-field partition function and spontaneous magnetisation (see Maillard and Rammal (1985) and references therein) was of course confirmed.

References

Adler J, Enting I G and Privman V 1983 *J. Phys. A: Math. Gen.* **16** 1967-73

de Neef T and Enting I G 1977 *J. Phys. A: Math. Gen.* **10** 801-5

Dhar D and Maillard J-M 1985 *J. Phys. A: Math. Gen.* **18** L383-8

Enting I G 1974 *J. Phys. A: Math., Nucl. Gen.* **7** 1617-26

——1978a *J. Phys. A: Math. Gen.* **11** 563-8

——1978b *Aust. J. Phys.* **31** 515-22

——1980a *J. Phys. A: Math. Gen.* **13** 3713-22

——1980b *J. Phys. A: Math. Gen.* **13** L133-6

——1980c *J. Phys. A: Math. Gen.* **13** L409-12

Enting I G and Wu F Y 1982 *J. Stat. Phys.* **28** 351-73

Jaekel M T and Maillard J M 1984 *J. Phys. A: Math. Gen.* **17** 2079-94

Maillard J M and Rammal R 1985 *J. Phys. A: Math. Gen.* **18** 833-46

Wortis M 1974 *Phase Transitions and Critical Phenomena* vol 3, ed C Domb and M S Green (New York: Academic) ch 3